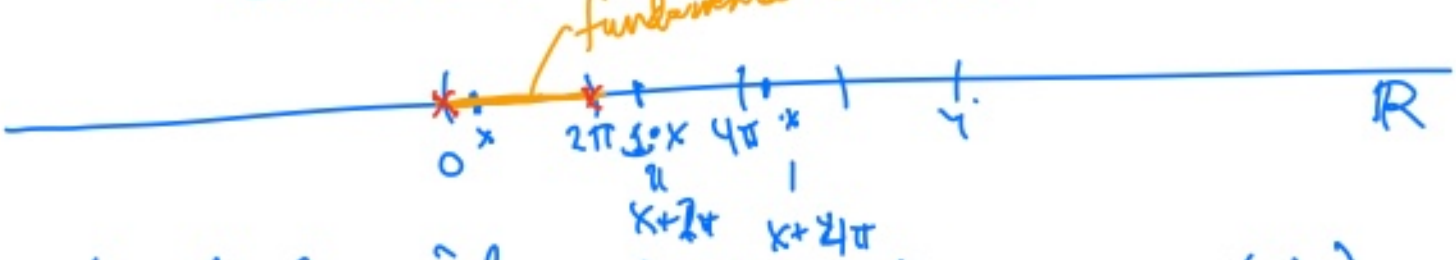
Series of talks — R-torsion.

$\mathbb{R}$ 1-dim'l manifold

Action: $\mathbb{Z}$ acts on $\mathbb{R}$

Given $x \in \mathbb{R}$, $m \in \mathbb{Z}$ $\quad m \cdot x = x + m2\pi$



The set of equivalence classes (i.e. orbits of this action) as a set. $[x] = \{x+2\pi, x-2\pi, x+4\pi, x-4\pi, \dots\}$

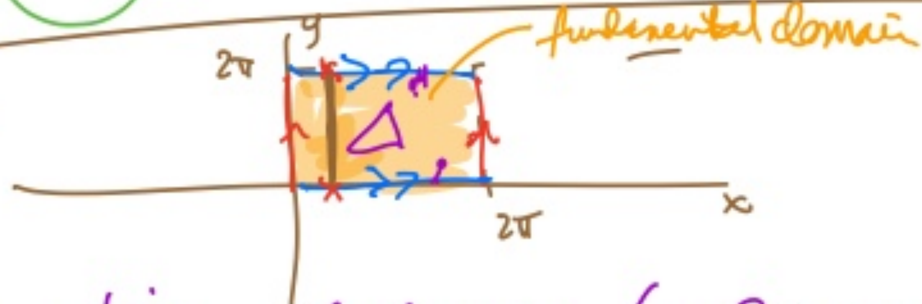
$\Rightarrow \mathbb{R}/\mathbb{Z} = \{[x]\} =$

$\cong$

S^1

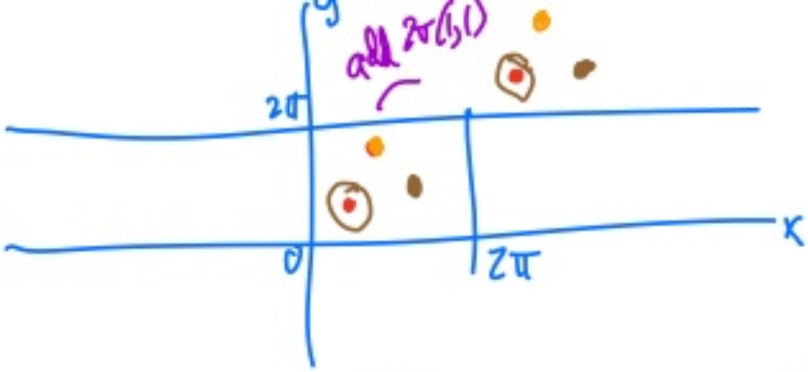
Circle

$T^2 = \mathbb{R}^2 / 2\pi\mathbb{Z}^2$

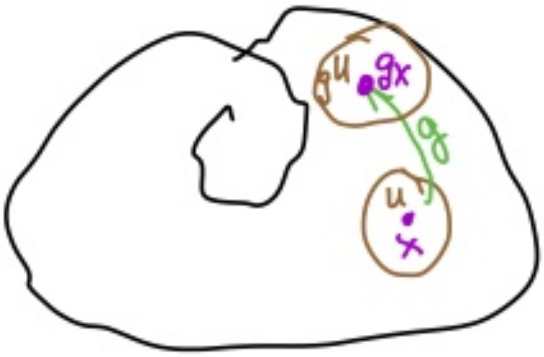


group action $(x,y) \rightsquigarrow (x + 2\pi m, y + 2\pi q)$ where $m, q \in \mathbb{Z}$.

Both of these group actions on $\mathbb{R}$ & $\mathbb{R}^2$ are actions by isometries — length & angle-preserving.



These group actions are <u>properly discontinuous actions</u>. G acting on M



$$\boxed{\forall x \in M,\ \exists \text{ nbhd } U \text{ containing } x \text{ such that } \forall g \in G \quad U \cap gU = \emptyset.}$$

<u>Thm</u> If M is a connected & simply connected manifold & G acts on M properly discontinuously, then

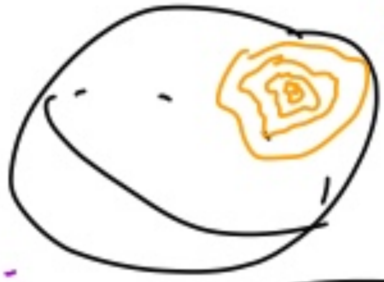
$\pi_1(M/G) =$ "fundamental group of M/G"
$\cong G$

This is a topological invariant.

$\mathbb{R}$ & $\mathbb{R}^2$ & $\mathbb{R}^4$ are simply connected.

simply connected — <u>every</u> closed curve on the space can be <u>continuously</u> contracted to a point.

Simply connected.



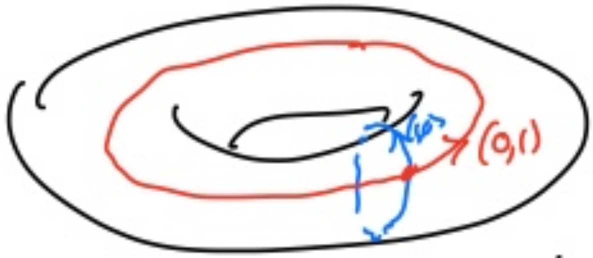
Not simply connected

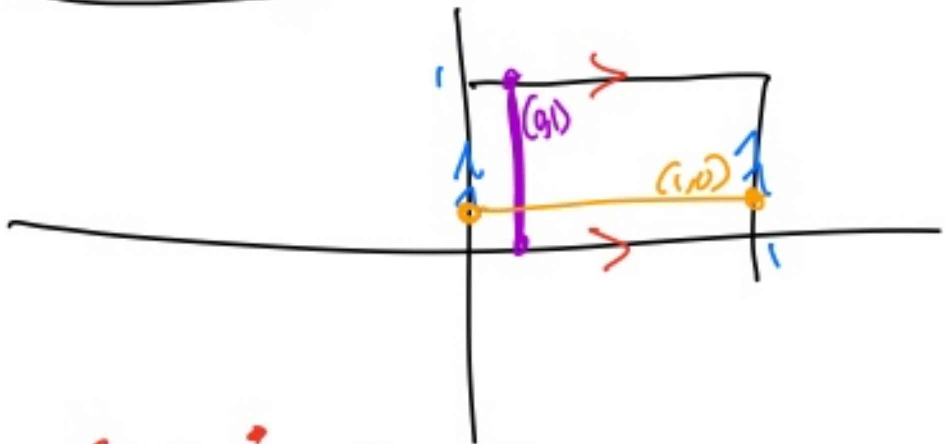
can't be contracted to a point

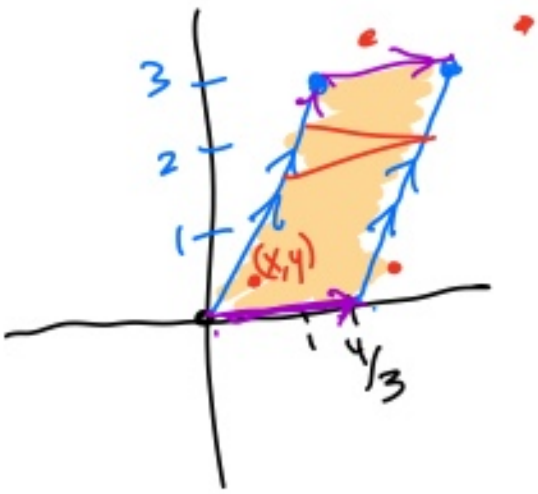
$$T^2 = \mathbb{R}^2 / 2\pi\mathbb{Z}^2 \Rightarrow \pi_1(T^2) \cong \mathbb{Z}^2 \cong 2\pi\mathbb{Z}^2.$$

s.conn.

generators
(1,0)
(0,1)





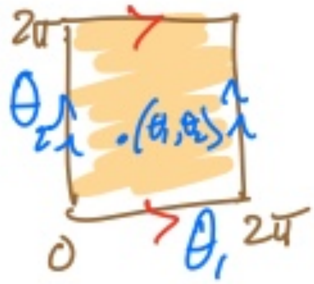


$\mathbb{R}^2 / G$ a torus.

G = group generated by

$(1,3)$, $(\frac{4}{3}, 0)$ (vector addition)

$$(x,y) + m(1,3) + n(\tfrac{4}{3}, 0)$$

Embedding  into $\mathbb{R}^4$.

$$F(\theta_1,\theta_2) = \big(\underset{\text{circle}}{e^{i\theta_1}}, \underset{\text{circle}}{e^{i\theta_2}}\big) \in \mathbb{C}^2$$

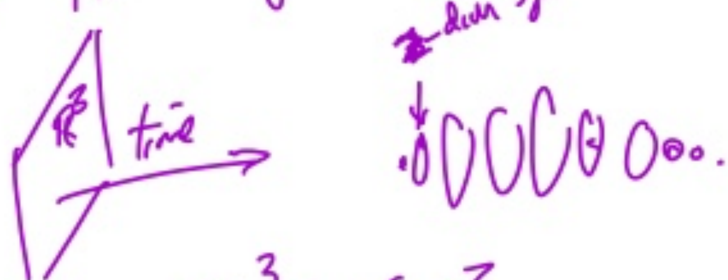
$$= \underline{(\cos\theta_1, \sin\theta_1, \cos\theta_2, \sin\theta_2)} \in \mathbb{R}^4$$

$$S^3 = \{(x_1,x_2,x_3,x_4) \in \mathbb{R}^4 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

$$= \{(z_1,z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

$$= \{q \in \mathbb{H} : |q| = 1.\}$$

How to think of S^3

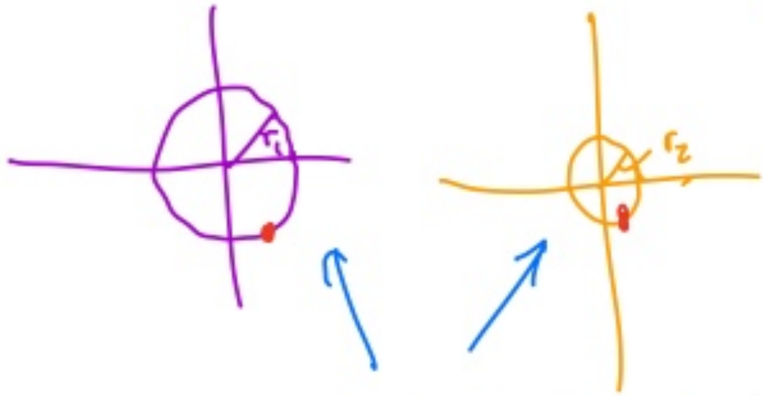


$$S^3 = \mathbb{R}^3 \cup \{\infty\}$$

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$$

$|z_1| = r_1$, $|z_2| = r_2$

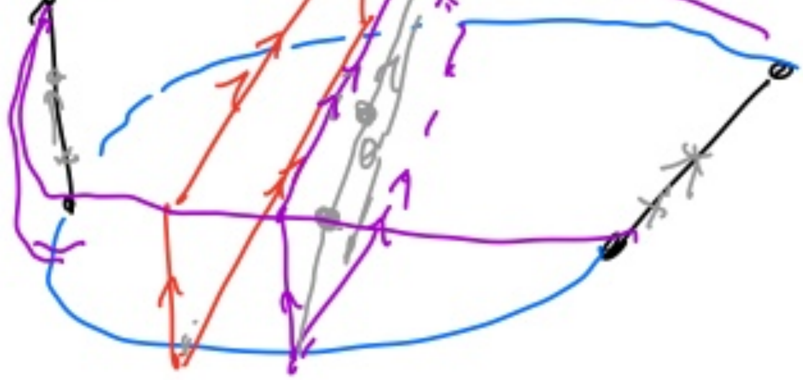
$r_1^2 + r_2^2 = 1$



a T^2 inside $\mathbb{C}^2$.

S^3 = union of tori (radii (r_1, r_2) on unit circle.

(and 2 circles $r_1 = 0, r_2 = 1$
$r_1 = 1, r_2 = 0$)



Another way to think of S^3.

A group action on S^3.

Given (z_1, z_2) on S^3 $\quad (|z_1|^2 + |z_2|^2 = 1)$

Given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, let

$$\phi(\theta)(z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$$

Notice

$$\phi : \underset{\substack{\| \\ S^1 = \{\theta\}}}{G} \to \begin{pmatrix} \text{isometries} \\ \text{of } \mathbb{C}^2 \end{pmatrix}.$$

$$|e^{i\theta} z_1| = |e^{i\theta}|\,|z_1| = |z_1|$$

$$|e^{i\theta} z_2| = |z_2|.$$

It maps the torus inside S^3 to itself. $\longrightarrow$ maps S^3 to itself.

Every "orbit" of this group action is a circle.

Quotient $\quad S^3/S^1 \cong S^2$

$\uparrow$ can prove this is the same.

Called the Hopf Fibration.

Finite group action on S^3

$\mathbb{Z}_p = \{\text{integers mod } p\}$ $(p \in \mathbb{N})$

Let $q \in \mathbb{N}$ s.t. $\gcd(p,q)=1$ — q is relatively prime to p

The Lens space is

$$S^3/\mathbb{Z}_p = \{(z_1,z_2)\in S^3\}/\text{action.}$$

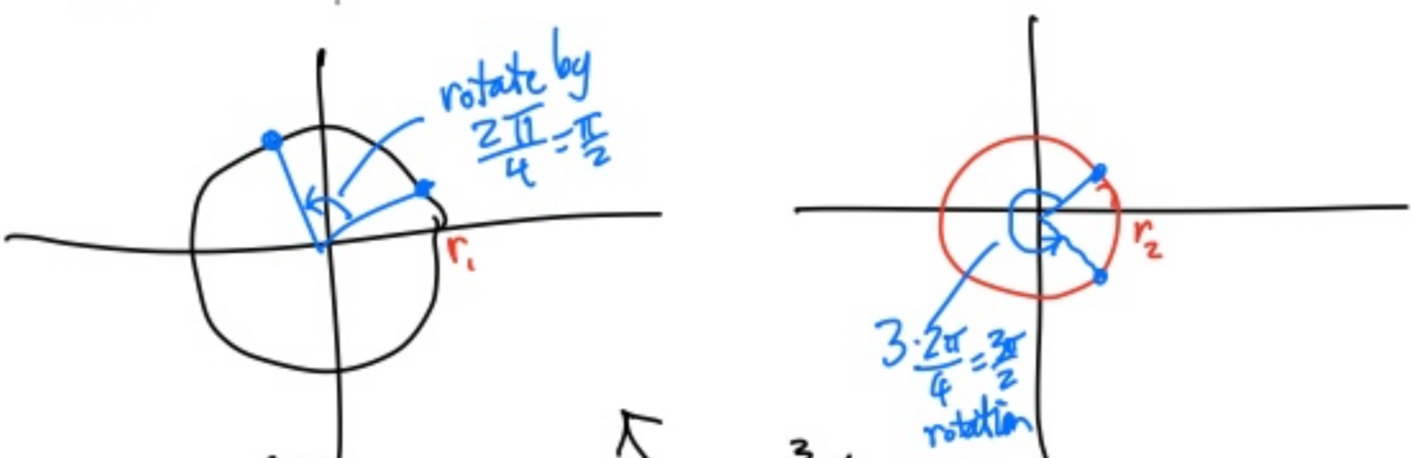
This is a properly discontinuous action.

Take $a \in \mathbb{Z}_p$

$$a\cdot(z_1,z_2) = \left(e^{i\frac{2\pi a}{p}} z_1, e^{i\frac{2\pi q a}{p}} z_2\right)$$

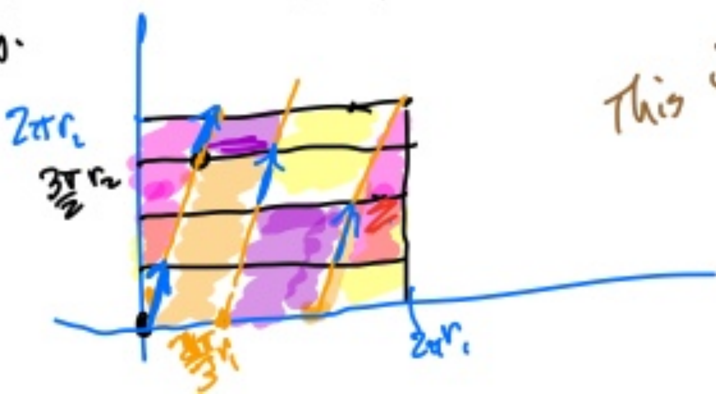
(This is actually an isometric action.)

Example: $p=4$ $q=3$

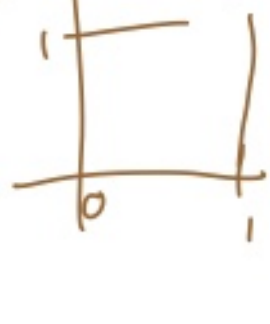


What does this look like on each torus.

$L(4,3) = S^3/\mathbb{Z}_4$



This is like $T^2 = S^1\times S^1 \bmod \left(\frac{1}{4}, \frac{3}{4}\right)$



Big Questions about $L(p,q)$:

Defined by Tietze 1908 "torus manifolds"

"lens spaces" Seifert & Threlfall.